A first step into Nonlinear Statics

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Abstract
In the introductory courses in Statics and Structural Mechanics, the role played by nonlinearities is often left out, to make more room to details in linear problems, which can be faced with relatively simple mathematical tools. Behavior of structures is appropriately modeled as linear in many circumstances, although we have to take nonlinearities into account whenever stresses or strains become large enough, or members have very small thickness or even very large slenderness, and so on. Hence, we could say that behavior of structures is inherently nonlinear, and linear approximation is valid provided specific requirements on the order of magnitude of given ingredients of the problem under consideration are fulfilled. The scope of this brief contribution is reveal, through a paradigmatic example, how nonlinearity naturally emerges.

Introduction
Highly deformable structures made assembling very thin rods or membranes are used in many applications in architecture and engineering.It is noteworthy that thin members can sustain very small or even vanishing moments. Hence, rods and membranes should be properly designed to respectively undergo axial and in-plane stresses only. However thin structures are particularly prone to suffer buckling instability and are very responsive to imperfections.

Besides these aspects, which could be, in some sense, considered features of highly deformable structures, another key point is simply the fact that they significantly change their shape. In this respect, structures change the structural geometry passively, in order to accommodate external loads, or actively, to accomplish a specific task or perform actions, which the structures are programmed for. This latter is, for instance, the case of morphing structures, which can be designed to radically change their shape by exploiting several actuation devices [1]. Furthermore, structures working in a extremely deformed regime are found in many every-day-use devices [2].
As a results, the underlying problem is inherently nonlinear and both the static and dynamic analysis of highly deformable structures become significantly more challenging than that of structures and buildings that can be modeled as linear, at least if some specific conditions are met.
While the cited aspects should sound quite obvious to any person with a major education in Mechanics, whatever theoretical or applied, it could produce some ambiguity for students in architecture and engineering.
Indeed, difficulties arising from many nonlinear problems are such that their descriptions are rarely taught to undergraduate students [3]. So, when they are taught basic Mechanics and, in particular, the laws of the balance of bodies and structures,it is said that if the deformed configuration of a system is close to the initial, undeformed one, the latter can be adopted in writing equilibrium equations. Of course, such an approximation introduces errors, which become smaller and smaller as close the deformed configuration is to the initial one. While this is a price to pay, on the other hand, the related advantages are considerable. In fact, deformed configuration is not known in advance while the initial state, which is conventionally assumed stress-free, is given. Such a condition is generally fulfilled by deformable systems and structures undergoing small displacements and rotations, accompanied by small strains.
On the contrary, also in case of small strains, highly deformable structures exhibit large displacements and rotations, leading to equilibrium equations which must be mandatory written on the unknown current configuration. Hence, such a problem is geometrically nonlinear. Besides this kind on nonlinearity, many others do exist, and thus the behavior of real structures is, in general, nonlinear, due to many different reasons.
While a comprehensive description of general nonlinearities is certainly beyond the scope of this contribution, which is aimed at being of didactic use, we recall that lack in linearity relevant for structures of specific interest in architecture and civil engineering can be roughly grouped in two broad sets, respectively called as the mechanical and geometrical nonlinearity.

The former refers to not-proportional relationship between strains and stresses, dependence on load history, possible presence of permanent deformations, different behaviors in tension and compression, and so on. The latter considers cases in which the changes in the shape of the structure are not negligible, remaining the constitutive assumption essentially linear.
Although very broad, such a kind of two-sets classification may be however unable to cover all the possible nonlinearities. We refer to [4,5] where concepts of geometric nonlinearity, stability theory and plasticity are explored in detail, or to [6] for a discussion about the main nonlinearities related to Solid Mechanics, including mechanical and contact nonlinearities. Furthermore, there is a large amount of literature concerning nonlinear problems. Besides the already cited literature sources, as suggestions for further readings, at different levels of difficulties and technicalities, we cite some books, among the many existing.
For a comprehensive treatment on one-dimensional structural members, as rods and beams, or two-dimensional bodies, as plates, shells and membranes, we refer to [7], where linear and nonlinear theories are reported. Beams and cables are extensively discussed in [8] and [9], this latter discussing in depth about the particular and technically relevant case of composite beams. Thin-walled elastic shells are considered in [10].
For a theoretical framework of Nonlinear Mechanics, along with computational methods, applications and parametric investigations of nonlinear phenomena refer to [11] and [12], the former giving also mechanical interpretation towards design, the latter analyzing nonlinear elastic and elastoplastic materials, including bifurcation and instability. For further detail on materials in nonlinear regimes, see also [13].
Analysis of nonlinear behavior of solids and structures suitable for numerical computation is presented in [14] and [15], this latter mainly devoted to elastoplastic finite element procedures, restricting the attention to one-dimensional plasticity.
Finally, in order to emphasize some basic, though key, features of nonlinear problems in Statics, we consider, in what follow, a simple one-dimensional example, which can be seen as a prototypical idealization of a thin, deformable string subjected to a transverse point load in its midspan cross section.

A benchmark example of Nonlinear Statics
Let us consider the mechanical system, shown in fig.01, made of two collinear, deformable rods, of initial length L, connected, mutually and to the ground, by hinges, and assume the central node is subjected to a static point load of intensity $2P$. A variant, mechanically equivalent, of such a structure in analyzed in [6]. It is noteworthy that the system satisfies necessary conditions to be isostatic, but

however it can also be proved that the matrix of coefficients associated to the equilibrium equations, written with respect to (w.r.t.) the straight configuration, is singular.

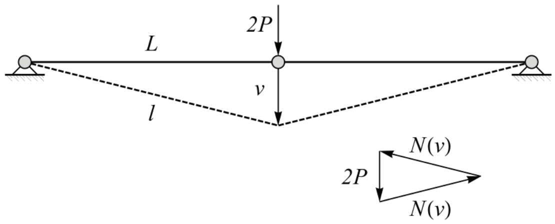

fig.01 - A two-bar mechanical system. The bars are assumed to be extensible.

As a consequence, the central node is allowed to infinitesimally move along the direction orthogonal to the rods. In terms of equilibrium, in the linear theory, the system cannot sustain the assigned load, with finite axial forces along the bars. Let us now consider the deformed configuration, reported in fig.01 as a dashed line, where the mid-point has an unknown finite displacement v, with the straight bars stretched at a current length l such that

$$l = \sqrt{L^2 + v^2}\,. \qquad (1)$$

By writing the equilibrium w.r.t. the current configuration, the solution is achieved in terms of v, and thus also the axial force along the bars can be written as function of v, that is $N = N(v)$. Since bars rotate, clockwise or counterclockwise, of the angle $\varphi + \varphi(v)$ defined as

$$\varphi = \arctan\left(\frac{v}{L}\right), \qquad (2)$$

the value of the axial force is computed as

$$N = \frac{P}{\sin\varphi} = \frac{P\,l}{v}, \qquad (3)$$

as also graphically shown in the force polygon in fig.01. By virtue of Eq. (3), it is simple to recognize that, for finite P and vanishing v, N becomes infinite. On the opposite, that is for $v \to \infty$, $N \to P$: indeed, in such a case, the direction of the two bars, both experiencing infinite elongation, coincides with the direction of the point load $2\,P$. The graph of N, rescaled by P, as a function of the dimensionless displacement

$$\bar{v} = \frac{v}{L}, \qquad (4)$$

is shown in fig.02. Of course, both limit situations are physically meaningless, the former implying an infinite internal force, the latter an infinite deformation, both not attainable in any real bar, whatever the size of the member, or the material it is made of. Finally, let us notice that the structure reported in fig.01 can be considered a suitable model for a deformable string subjected to a transverse point load in its midspan cross section. The analogy between the two problems stems from the fact that both cannot sustain transverse shearing forces and bending moments. Furthermore, at the equilibrium, the string must exhibit not-vanishing, concentrated curvature only where the load is applied, being straight and under tensile stress the rest of the string.

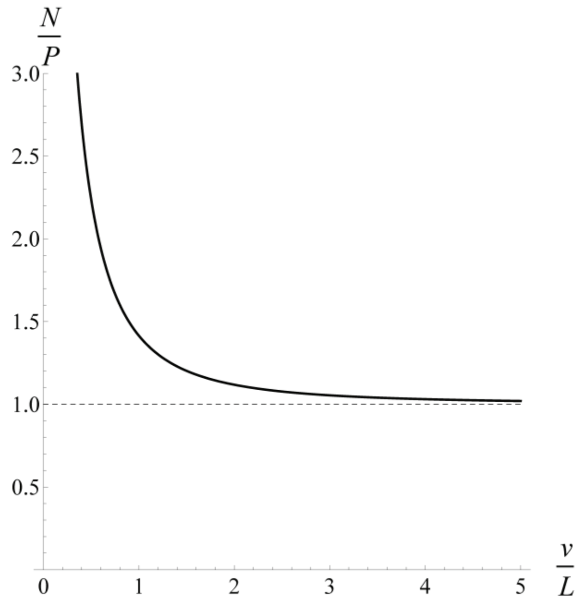

fig.02 - The variation of the dimensionless axial force as a function of the dimensionless displacement.

Strain measures

We may introduce several strains. On introducing the Lagrangian strain

$$\lambda = \frac{l}{L}\,, \qquad (5)$$

among others, we adopt hereafter three strain measures, as

$$\varepsilon = \lambda - 1\,, \qquad e = \frac{1}{2}(\lambda^2 - 1)\,, \qquad s = \log\lambda\,. \qquad (6)$$

The strain ε, the engineering strain usually applied in linear theory, simply compares how much a bar has been extended w.r.t. its reference length, and remarkably it measures the strain along the rotating bar. Indeed it is a rotated strain and gives accurate results provided the deformation of the body is small enough. In case of finite deformation, the finite Green's strains e or the logarithmic strain s, as the true strain, are often used.

Constitutive assumptions

Let us consider a generic material with a linear relationship between the generalized stress N and the measures of strain defined in Eqs. (6), that is

$$N_\varepsilon = K\varepsilon \qquad N_e = Ke \qquad N_s = Ks \qquad (7)$$

assuming here and henceforth that the stiffness K, that is the Young's modulus E times the cross-sectional area A, is independent of the state of stress, on the temperature and so on, in order to make computations as simple as possible. In fig.03, the three versions of N, given by Eqs. (7), rescaled by K, are drawn as functions of λ. It is evident that, for small values of λ, Eqs. (7) give close responses, leading to see also that the three different strain measures are equivalent to each other, provided displacement v remains small enough. However, forces and strains become significantly different for large displacements. Furthermore, for extreme shortening, both engineering strain and Green's strain give clearly wrong predictions, allowing finite force even in the limit of λ approaching zero. On the contrary, the strain s, leads to increasing compressive force as λ decreases.

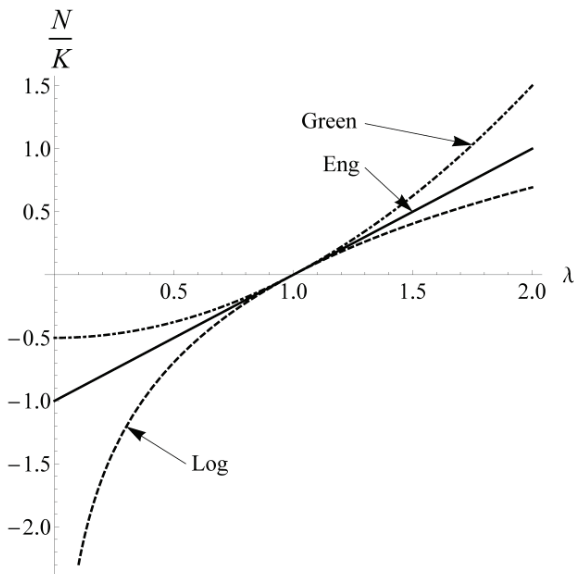

fig.03 - Graphs of three constitutive relationships, one linear (black solid line), the other two nonlinear (dashed line for the constitutive assumption with logarithmic strain, dotted-dashed line for that with finite Green's strain).

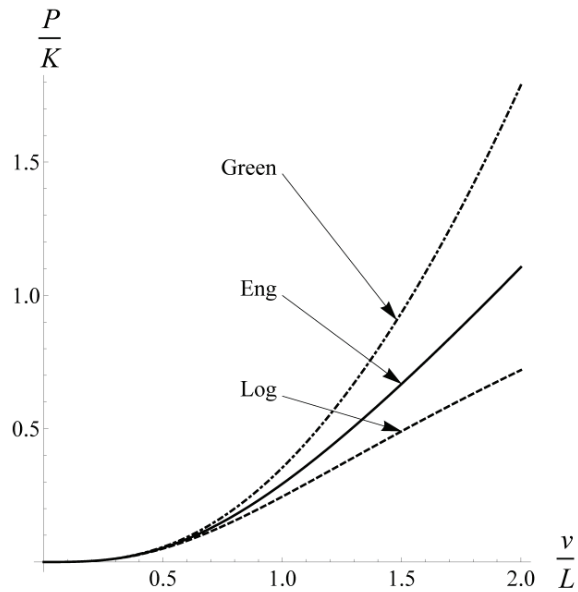

fig.04 - Nonlinear load-displacement curves: black solid line, dashed line and dotted-dashed line correspond to the engineering strain, logarithmic strain and finite Green's strain, respectively. Observe that also the curve corresponding to the linear strain is nonlinear, due to the geometrical nonlinearity of the problem under investigation.

Load-deflection curves

From Eqs. (3) and (7), taking into account the rotated engineering, Green's and the true strains, given by Eqs. (6), we attain

$$P_\varepsilon = 2\,K\alpha\,(\sqrt{1+\bar{v}^2}-1)\,,$$
$$P_e = K\alpha\,\bar{v}^2\,, \tag{8}$$
$$N_s = K\alpha\log\,(1+\bar{v}^2)\,,$$

being α a nonlinear function of $\bar{v}$ defined as

$$\alpha = \frac{\bar{v}}{2\sqrt{1+\bar{v}^2}}, \tag{9}$$

and $\bar{v}$ the dimensionless displacement defined in Eq. (4).

The loads given by Eqs. (8), rescaled by the stiffness K, are reported in fig.04, from which it can be seen that also load-displacement curves remain very close to each other for small displacements, and gives different predictions only for larger ones, as it happens for the constitutive relationships of the axial force N. This, among others, justifies the fact that the engineering strain, which is linear in λ, is conveniently used in all those applications showing a behavior adequately approximated as linear. However, a further remark is in order.

As already stated, in the example we are dealing with, the linear approximation suffers drawbacks. Now we see the reason.
Indeed, at zero displacement, the load-deflection curve has a horizontal tangent, implying that the example under consideration simply cannot be formulated in terms of a linearized theory.

Conclusions

In many applications in architecture and engineering, structural behavior may be modeled as linear, provided specific requirements are fulfilled.
However, nonlinear behavior of materials and structures is exploited to accomplish defined tasks. The behavior of highly deformable structures is inherently geometrically nonlinear, although the constitutive assumptions between stress and strain is still linear. However, under severe deformations, constitutive assumptions are nonlinear too.
The present contribution, which is a not general, nor complete survey in nonlinear problems, is intended as an invitation to learn about nonlinear theories.
In doing this, we reported about a simple, paradigmatic nonlinear example, which cannot be formulated in terms of a linearized theory and sheds light some basic features of nonlinear problems in Statics.

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